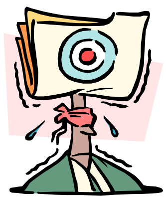
Induction



Overview

- Recursion
  - a strategy for writing programs that compute in a "divide-and-conquer" fashion
  - solve a large problem by breaking it up into smaller problems of same kind
- Induction
  - a mathematical strategy for proving statements about integers (more generally, about sets that can be ordered in some fairly general ways)
- Understanding induction is useful for figuring out how to write recursive code.

Defining Functions

- It is often useful to write a given function in different ways.
  - (eg) Let S:int → int be a function where S(n) is the sum of the natural numbers from 0 to n.

    S(0) = 0, S(3) = 0+1+2+3 = 6
  - One definition: iterative form
    - S(n) = 0+1+…+n
  - Another definition: closed-form
    - S(n) = n(n+1)/2

Equality of function definitions

- How would you prove the two definitions of S(n) are equal?
  - In this case, we can use fact that terms of series form an arithmetic progression.
- Unfortunately, this is not a very general proof strategy, and it fails for more complex (and more interesting) functions.

Sum of Squares Functions

- Here is a more complex example.
  - (eg) Let SQ:int →int be a function where SQ(n) is the sum of the squares of natural numbers from 0 to n.

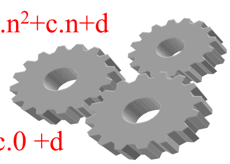
    $SQ(0) = 0$, $SQ(3) = 0^2+1^2+2^2+3^2 = 14$
- One definition:
  - $SQ(n) = 0^2+1^2+\ldots+n^2$
- Is there a closed-form expression for SQ(n)?

Closed-form expression for SQ(n)

- Sum of natural numbers up to n was n(n+1)/2 which is a quadratic in n.
- Inspired guess: perhaps sum of squares on natural numbers up to n is a cubic in n.
- So conjecture: $SQ(n) = a.n^3+b.n^2+c.n+d$ where a,b,c,d are unknown coefficients.
- How can we find the values of the four unknowns?
  - Use any 4 values of n to generate 4 linear equations, and solve.

Finding coefficients

$SQ(n) = 0^2+1^2+\ldots+n^2 = a.n^3+b.n^2+c.n+d$



- Let us use n=0,1,2,3.
- $SQ(0) = 0 = a.0 + b.0 + c.0 + d$
- $SQ(1) = 1 = a.1 + b.1 + c.1 + d$
- $SQ(2) = 5 = a.8 + b.4 + c.2 + d$
- $SQ(3) = 14 = a.27 + b.9 + c.3 + d$
- Solve these 4 equations to get

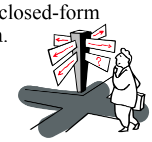
  a = 1/3, b = ½, c = 1/6, d = 0

- This suggests

  $SQ(n) \equiv 0^2+1^2+\ldots+n^2$

  $= n^3/3 + n^2/2 + n/6$

  $= n(n+1)(2n+1)/6$
- Question: How do we know this closed-form solution is true for all values of n?
  - Remember, we only used n = 0,1,2,3 to determine these co-efficients. We do not know that the closed-form expression is valid for other values of n.



- One approach:
  - Try a few values of n to see if they work.
  - Try n = 5. SQ(n) = 0+1+4+9+16+25 = 55
  - Closed-form expression: 5*6*11/6 = 55
  - Works!
  - Try some more values….
- Problem: we can never prove validity of closed-form solution for all values of n this way since there are an infinite number of values of n.

To solve this problem, let us express SQ(n) in another way.

$$SQ(n) = \underbrace{0^2 + 1^2 + \ldots + (n-1)^2}_{SQ(n-1)} + n^2$$

This leads to the following recursive definition of SQ:

$SQ(0) = 0$

$SQ(n) = SQ(n-1) + n^2 \mid n > 0$

Vertical bar | means "whenever"

To get a feel for this definition, let us look at

$SQ(4) = SQ(3) + 4^2 = SQ(2) + 3^2 + 4^2 = SQ(1) + 2^2 + 3^2 + 4^2$

$= SQ(0) + 1^2 + 2^2 + 3^2 + 4^2 = 0 + 1^2 + 2^2 + 3^2 + 4^2$

Notation for recursive functions

Base case

$SQ(0) = 0$

$SQ(n) = SQ(n-1) + n^2 \mid n > 0$

Recursive case

Can we show that these two definitions of SQ(n) are equal?

$SQ_r(0) = 0$

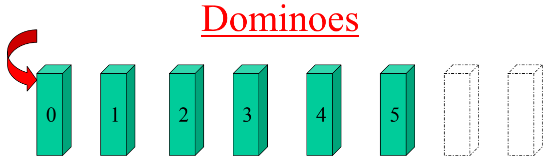
$SQ_r(n) = SQ_r(n-1) + n^2 \mid n > 0$

r: recursive

$SQ_c(n) = n(n+1)(2n+1)/6$

c: closed-form

Dominoes



- Assume equally spaced dominoes, and assume that spacing between dominoes is less than domino length.
- How would you argue that all dominoes would fall?
- Dumb argument:
  - Domino 0 falls because we push it over.
  - Domino 1 falls because domino 0 falls, domino 0 is longer than inter-domino spacing, so it knocks over domino 1.
  - Domino 2 falls because domino 1 falls, domino 1 is longer than inter-domino spacing, so it knocks over domino 2.
  - ……
- Is there a more compact argument we can make?

Better argument

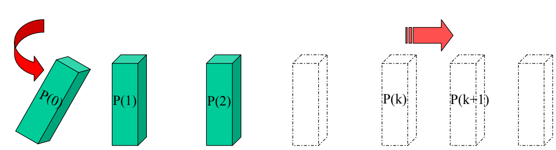
- Argument:
  - Domino 0 falls because we push it over (base case).
  - Assume that domino k falls over (inductive hypothesis).
  - Because domino k's length is larger than inter-domino spacing, it will knock over domino k+1 (inductive step).
  - Because we could have picked any domino to be the k^{th} one, we conclude that all dominoes will fall over (conclusion).
- This is an inductive argument.
- This is called weak induction. There is also strong induction (see later).
- Not only is it more compact, but it works even for an infinite number of dominoes!

Weak induction over integers

- We want to prove that some property P holds for all integers $n \geq 0$.
- Inductive argument:
  - P(0): (base case) show that property P is true for 0
  - P(k): (inductive hypothesis) assume that P(k) is true for a particular integer k.
  - P(k) => P(k+1): (inductive step) show that if property P is true for integer k, it is true for integer k+1
  - P(n): (conclusion) Because we could have picked any k, this means P(n) holds for all integers $n \geq 0$.

$SQ_r(n) = SQ_c(n)$ for all n?

Define P(n) as $SQ_r(n)= SQ_c(n)$



Prove P(0).

Assume P(k) for particular k.

Prove P(k+1) assuming P(k).

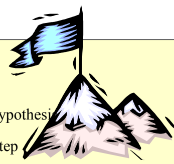
$SQ_r(0) = 0$

$SQ_r(n) = SQ_r(n-1) + n^2$

$SQ_c(n) = n(n+1)(2n+1)/6$

Let P(n) be the proposition that $SQ_r(n) = SQ_c(n)$.

Proof by induction:



P(0): show $SQ_r(0) = SQ_c(0)$ — Base case

(easy) $SQ_r(0) = 0 = SQ_c(0)$

Assume SQr(k) = SQc(k) — Inductive hypothesis

Prove that P(k) => P(k+1): — Inductive step

$SQ_r(k+1) = SQ_r(k) + (k+1)^2$	(definition of SQ_1)
$= SQ_c(k) + (k+1)^2$	(inductive hypothesis)
$= k(k+1)(2k+1)/6 + (k+1)^2$	(definition of SQ_2)
$=(k+1)(k+2)(2k+3)/6$	(algebra)
$= SQ_c(k+1)$	(definition of SQ_2)

Therefore, SQr(n) = SQc(n) for all integers n. Conclusion

Another example of weak induction

Prove that the sum of the first n integers is n(n+1)/2.

Let $S(i) = 0+1+2+\ldots+i$

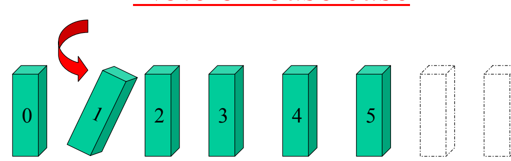
Show that $S(n) = n(n+1)/2$.

- Base case: $(n=0)$
  - $S(0) = 0$
- Inductive hypothesis:
  - Assume $S(k) = k(k+1)/2$ for a particular k.
- Inductive step:
  - $S(k+1) = 0 + 1 + \ldots + k + (k+1) = S(k) + (k+1)$
    $= k(k+1)/2 + (k+1)$
    $= (k+1)(k+2)/2$
  - Therefore, if result is true for k, it is true for $k+1$.
- Conclusion: result follows for all integers.
- Note: we did not use arithmetic progressions theory.

Essence of proof is the following recursive description of $S(k)$

$S(0) = 0$
$S(k) = S(k-1) + k \mid k>0$

Note on base case



- In some problems, we are interested in showing some proposition is true for integers greater than or equal to some lower bound (say b)
- Intuition: we knock over domino b, and dominoes in front get knocked over. Not interested in dominoes 0,1,…,(b-1).
- In general, base case in induction does not have to be 0.
- If base case is some integer b, induction proves proposition for $n = b, b+1, b+2, \ldots$.
- Does not say anything about $n = 0, 1, \ldots, b-1$

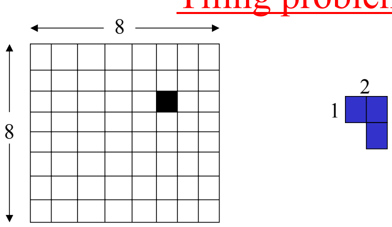
Weak induction: non-zero base case

- We want to prove that some property P holds for all integers $n \geq b$
- Inductive argument:
  - P(b): show that property P is true for integer b
  - P(k): assume that P(k) is true for a particular integer k.
  - P(k) => P(k+1): show that if property P is true for integer k, it is true for integer k+1
  - P(n): Because we could have picked any k, this means P(n) holds for all integers $n \geq b$.

More on induction

- In some problems, it may be tricky to determine how to set up the induction:
  - What are the dominoes?
- This is particularly true in geometric problems that can be attacked using induction.

Tiling problem

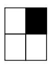


- Problem:
  - A chess-board has one square cut out of it.
  - Can the remaining board be tiled using tiles of the shape shown in the picture?
- Not obvious that we can use induction to solve this problem.

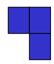
Idea

- Consider boards of size 2^n x 2^n for n = 1,2,…..
- Base case: show that tiling is possible for 2 x 2 board.
- Inductive hypothesis: assume 2^k x 2^k board can be tiled
- Inductive step: assuming 2^k x 2^k board can be tiled, show that 2^{k+1} x 2^{k+1} board can be tiled.
- Draw conclusion
  - Chess-board (8x8) is a special case of this argument
  - We have proved special case of chess-board by proving generalized problem!

Base case

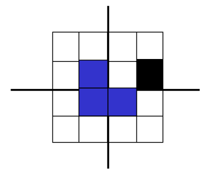


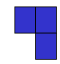
2x2 board



- For a 2x2 board, it is is trivial to tile the board regardless of which one of the four pieces has been cut.

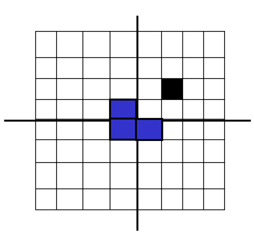
4x4 case

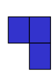




- Divide 4x4 board into four 2x2 sub-boards.
- One of the four sub-boards has the missing piece.
- That sub-board can be tiled since it is a 2x2 board with a missing piece.
- Tile the center squares of the three remaining sub-boards as shown.
- This leaves 3 2x2 boards with a missing piece, which can be tiled.

8x8 case





- Divide board into 4 sub-boards and tile the center squares of the three complete sub-boards.
- The remaining portions of the 4 sub-boards can be tiled by assumption about 4x4 boards.

Inductive proof

- Claim: Any board of size $2^n \times 2^n$ with one missing square can be tiled.
- Proof: by induction.
  - Base case: (n = 1) trivial since board with missing piece is isomorphic to tile.
  - Inductive hypothesis: board of size $2^k \times 2^k$ can be tiled
  - Inductive step: consider board of size $2^{k+1} \times 2^{k+1}$
    - Divide board into four equal sub-boards of size $2^k \times 2^k$
    - One of the sub-boards has the missing piece; by inductive hypothesis, this can be tiled.
    - Tile the central squares of the remaining three sub-boards as discussed before.
    - This leaves three sub-boards with a missing square each, which can be tiled by inductive hypothesis.
  - Conclusion: any board of size $2^n \times 2^n$ with one missing square can be tiled.

When induction fails

- Sometimes, an inductive proof strategy for some proposition may fail.
- This does not necessarily mean that the proposition is wrong.
  - It just means that the inductive strategy you are trying fails.
- A different induction or a different proof strategy altogether may succeed.

Tiling example (contd.)

- Let us try a different inductive strategy which will fail.
- Proposition: any *n x n* board with one missing square can be tiled.
- Problem: a *3 x 3* board with one missing square has 8 remaining squares, but our tile has 3 squares. Tiling is impossible.
- Therefore, any attempt to give an inductive proof is proposition must fail.
- This does not say anything about the 8x8 case.

Strong induction

- We want to prove that some property P holds for all integers.
- Weak induction:
  - P(0): show that property P is true for integer 0
  - Assume P(k) for a particular integer k.
  - P(k) => P(k+1): show that if property P is true for integer k, it is true for k+1
  - Conclude that P(n) holds for all integers n.
- Strong induction:
  - P(0): show that property P is true for integer 0
  - Assume P(0) and P(1) …and P(k) for particular k.
  - P(0) and P(1) and …and P(k) => P(k+1): show that if P is true for integers less than or equal to k, it is true for k+1
  - Conclude that P(n) holds for all integers n.
- For our purpose, both proof techniques are equally powerful.

Editorial comments

- Induction is a powerful technique for proving propositions.
- We used recursive definition of functions as a step towards formulating inductive proofs.
- However, recursion is useful in its own right.
- There are closed-form expressions for sum of cubes of natural numbers, sum of fourth powers etc. (see any book on number theory).