# CS211, Lecture 27
# More Algorithms

## Announcements:

- course almost done: L28 is summary, evals
- A6, A7 due soon
- end of regular consulting: Fri, 5/1
- special hours to finish regrades, pick up work will be announced on the website
- consulting: forms in 303 Upson, starting 1PM today
- final exam info: Final Exam (on website)
  (review info, prior exam posted soon)
- final exam conflicts? see website; due Friday 5/2!

## Overview:

- shortest path algorithm for weighted graph
  (Dijkstra's algorithm)
- all pairs source shortest path
  (Floyd's algorithm)
- minimum cost spanning trees
  (Prim's algorithm, Kruskal's algorithm)

# 1. Shortest Path for Weighted Graphs

## 1.1 Assumptions

- could be directed or undirected
- non-negative weights

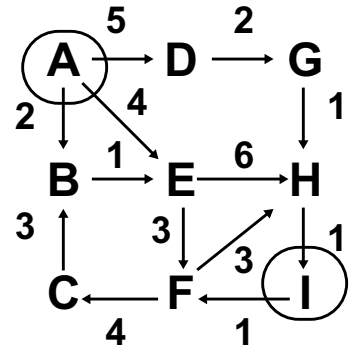
## 1.2 Dijkstra's Algorithm

- very famous
- example of greedy algorithm
- on-line demo:
  http://www-b2.is.tokushima-u.ac.jp/~ikeda/suuri/dijkstra/Dijkstra.shtml

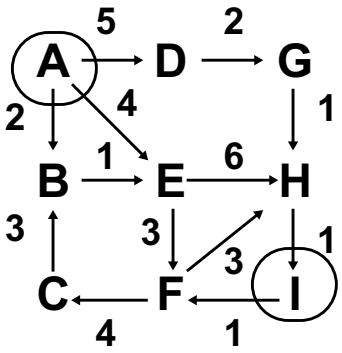
## 1.3 Wordy Gist: Based ON BFS

- BFS: visit the nodes by “levels” or “layers”
  - put new (unvisited) nodes in Q
  - look at each node at each layer
  - process each node and repeat
  - don’t re-process already-visited nodes
- New twist!
  - don’t treat all unvisited nodes as equals
  - want smallest accumulation of weights
  - so, need to sum weights along the way and maybe pick a different node than what’s in front of the Q

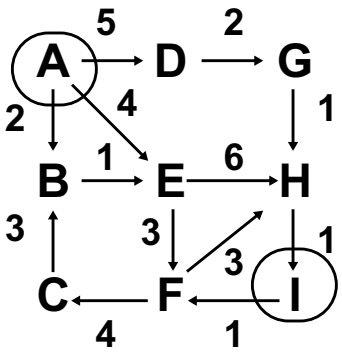
## 1.4 Physical Gist



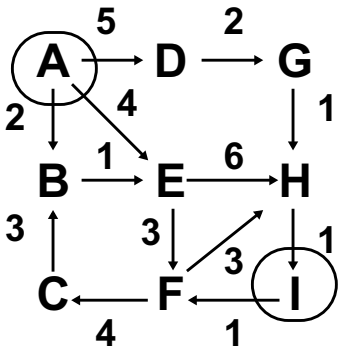

- Want shortest path from A to I
- Imagine graph is weights and strings, in which strings are cut to scaled lengths
- Pick up weights one at a time



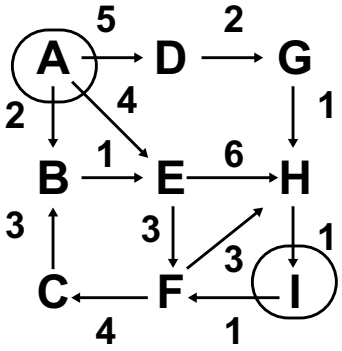

- Pick up A
- String becomes tight *first* at B
  record: A→B



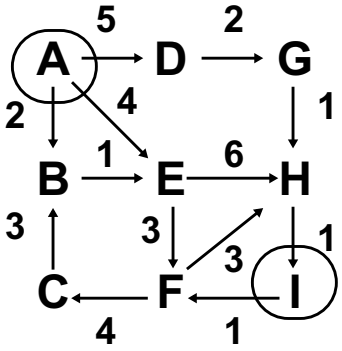

- Pick up B
- String becomes tight *first* at E
  record: A→B→E




- String now becomes tighter at D
- Why? After E comes F or H, each of which is longer than D
- record: A→D
- We could have gotten to E via A



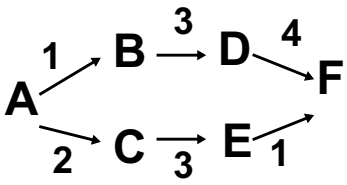

- Pick up D, followed by F
- But, G will be in "next round"
- so, record: A→D→G




- Eventually:
- record: A→D→G→H→I
- Forms a tree

## 1.5 Pseudocode Gist: Version 1

- Longish algorithm that uses cost in organizing priority queue to choose nodes
- a bit expanded on "wordy gist" from before:
  - pick the highest priority node (the smallest dist)
  - tag the node, record previous, update cost:
    PQ element: <node,accumulating cost>
  - repeat until no more PQ or no more unvisted nodes
    (note: tagging happens *after* extract from PQ)
- Visualization:




| Current PQ Entry | Current Node | Adjacent Nodes | PQ | Previous Node |
|---|---|---|---|---|
| <A, 0> | A | | [ ] | |
| | | B | [<B,1>] | A |
| | | C | [<B,1>,<C,2>] | A |
| <B,1> | B | | [<C,2>] | |
| | | D | [<C,2>,<D,4>] | A |
| <C,2> | C | | | |
| | | E | [<D,4>,<E,5>] | C |
| <D,4> | D | | | |
| | | F | [<E,5>,<F,8>] | D |
| <E,5> | E | | | |
| | | F | [<F,6>,<F,8>] | E |
| <F,6> | F | | | |
| | | | [<F,8>] | F |

## 1.6 Code Gist: Version 1

```
resetVerticies();
boolean done = false;
SeqStructure toDo = new Heap(edgeCount); // use min heap!
SeqStructure path = new QueueAsList();// should use stack
Vertex originVertex = (Vertex)verticies.get(origin);
Vertex endVertex = (Vertex)verticies.get(end);
originVertex.setPrev(null);
toDo.put(new MinPQElement(originVertex,0));
    while(!done && !toDo.isEmpty()) {
        MinPQElement entry = (MinPQElement) toDo.get();
        Vertex currentVertex = (Vertex) entry.getItem();

        if (!currentVertex.isVisited()) {
            currentVertex.visit();
            currentVertex.setCost(entry.getPriority());
            currentVertex.setPrev(
                        ((Vertex)entry.getItem()).getPrev());

            if (currentVertex.equals(endVertex))
                done = true;
            else {
                for (Iterator edges = currentVertex.
                            getEdgeIterator(); edges.hasNext(); ) {
                    Edge currentEdge = (Edge) edges.next();
                    Vertex nextVertex = currentEdge.getDest();

                    if(!nextVertex.isVisited()) {
                        int nextCost = currentEdge.getWeight() +
                                        currentVertex.getCost();
                        nextVertex.setCost(nextCost);
                        nextVertex.setPrev(currentVertex);
                        toDo.put(new
                                MinPQElement(nextVertex,nextCost));
                    }
                } // end for
            } // end else
        } // end if
    } // end while

path.put(endVertex);
while(endVertex.hasPrev()) {
    endVertex = endVertex.getPrev();
    path.put(endVertex);
}
return path;
```

## 1.7 Pseudocode Gist: Version 2

- Data:
  - s: start vertex
  - c(i,j): cost from i to j
  - dist(n): distance from s to n (initially ∞)
  - PQ to store neighboring nodes and choose the one with min cost at each "layer"
    (note: PQ size is edgeCount -> max # of adj nodes)
- Algorithm:

```
dist(s) <- 0
while (some vertices are unvisited)
   v <- unmarked vertex with smallest dist
        (get from the PQ)
   tag v
   for each node w adjacent to v
      dist(w) = min(dist(w),dist(v)+c(v,w))
   end for
end while
```

## 1.8 Code Gist: Version 2

```
public SeqStructure dijkstra3(Object origin,Object end) {

    resetVerticies(Integer.MAX_VALUE);
    SeqStructure toDo = new Heap(edgeCount);
    SeqStructure path = new QueueAsList();

    Vertex originVertex = (Vertex)verticies.get(origin);
    Vertex endVertex = (Vertex)verticies.get(end);

    originVertex.setPrev(null);
    originVertex.setCost(0);
    toDo.put(new MinPQElement(originVertex,0));

    while(!toDo.isEmpty()) {

        MinPQElement entry = (MinPQElement) toDo.get();
        Vertex currentVertex = (Vertex) entry.getItem();
        currentVertex.visit();

        for (Iterator edges = currentVertex.getEdgeIterator();
                                              edges.hasNext(); ) {

            Edge currentEdge = (Edge) edges.next();
            Vertex nextVertex = currentEdge.getDest();
            int nextCost =
                currentEdge.getWeight() + currentVertex.getCost();

            if (nextVertex.getCost() > nextCost ) {
                nextVertex.setCost(nextCost);
                nextVertex.setPrev(currentVertex);
                toDo.put(new MinPQElement(nextVertex,nextCost));
            }

        }

    }

    path.put(endVertex);
    while(endVertex.hasPrev()) {
        endVertex = endVertex.getPrev();
        path.put(endVertex);
    }
    return path;
}
```

## 1.9 Proof Gist

- Induction on iterations of while loop
  - each iteration moves one new node into lifted set
  - do induction on set of nodes ordered in the sequence in which they get put into the lifted set
- Induction:
  - base case: path from origin to self is 0
  - inductive hypothesis: assume that the shortest paths to all nodes currently in the lifted set have been computed correctly
  - inductive hypothesis: the next node that gets lifted is correct
- see Panels 16–19 at http://www.cs.cornell.edu/courses/cs211/2002sp/Lectures/graphs-quad.pdf

## 1.10 Run-time Analysis for Adjacency List

- dominant operation of method is while loop (processing unvisited nodes)
- time for processing each vertex:
  - each vertex processed once
  - all edges from a vertex might be processed
  - so, for each node, add up time for each edge
  - so, O(|V| + |E|) (see BFS time)
- PQ ops?
  - worst case: each edge has a node to queue and dequeue (see **`for`** loop and inner **`if`**)
  - so, PQ has max length of |E|
  - from heap: put is O(log n), get is O(log n)
  - so, adding each edge's contribution gives O(|E| log |E|)
- total: O(|V| + |E| log |E|)

## 1.11 Adjacency Matrix

- see DS&A pg 577
- O(|V|^2 + |E| log |E|)

# 2. All Pairs Shortest Path

## 2.1 Problem

- given edge weighted graph
- for each pair of verticies find length of shortest path

## 2.2 One Solution

- run Dijkstra's algorithm |V|+ times
- use each vertex as the origin

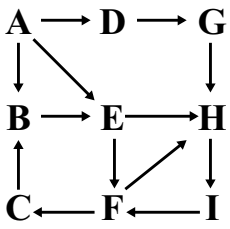
## 2.3 Floyd's Algorithm

- use adjacency matrix
- see 16.4.2 in DS&A

# 3. Spanning Trees

## 3.1 Interesting Thing About Traversals

- BFS, DFS don't repeat -> no cycles
- can backtrack to find a new unvisited node, but won't repeat it
- what does that look like?
- a rooted tree!
- **ex) BFS = {A,B,D,E,G,H,F,I,C}**



## 3.2 Spanning Tree

- effectively a subset of a graph:
  - all nodes sames as in G
  - tree edges must be graph edges (but nec all!)
  - connected
  - acyclic
- constructing?
  - pick a starting edge
  - add edges with unvisited dest nodes

## 3.3 Minimal Spanning Tree

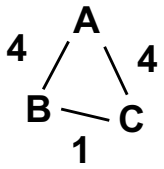
- given: undirected, weighted graph
- weight of spanning tree = sum of tree edge weights
- ***minimum spanning tree***:
  - any spanning tree with smallest weight
  - could have many such trees

## 3.4 Application

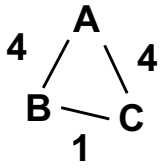
- see DS&SD pg 899
- find a cheap way to connect a bunch of nodes
  - as in something travelling an entire graph
  - plane needs to travel to a set of cities
  - wants cheapest path to take that still hits all cities

## 3.5 Compare to SSSP

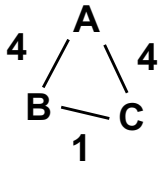
- SSSP: shortest path to a node
  what's cheapest way to get from A to Z using nodes {A,…,Z}
- MST: smallest sum of weights connecting each node
  what's cheapest way to connect all nodes {A,…,Z}?




**weighted, undirected graph**




**SSSP for A→C: {A,C}**
**Tree: { {A,B}, {A,C} }**




**MST for graph**
**Tree: { {A,B}, {B,C} }**

## 3.6 Prim's Algorithm

- modify Dijsktra's Algorithm:
  - put edges in PQ
  - associate edges with length of edge (don't add costs)
  - otherwise, algorithm is the same

## 3.7 Kruskal's Algorithm

- add edges by increasing order of weights
- not allowed to add edges that form cycles
- see DS&A 16.5.2

# 4. Exercises

- Modify the heap code to use a minimum heap.
- Modify the heap code to provide a sorted string for describing a priority queue.
- Prove by induction that Dijkstra's algorithm is correct.
- Implement Prim's algorithm.