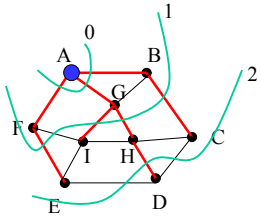
Minimal Spanning Trees

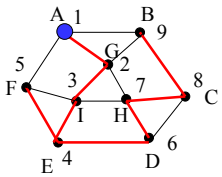
Spanning Tree

- Assume you have an undirected graph G = (V,E)
- Spanning tree of graph G is tree $T = (V, E_T \subseteq E, R)$
  - Tree has same set of nodes
  - All tree edges are graph edges
  - Root of tree is R

Spanning trees



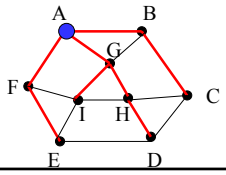
Breadth-first Spanning Tree



Depth-first spanning tree

Property 1 of spanning trees

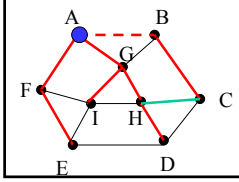
- Graph: G = (V,E)
- Spanning tree: $T = (V,E_T,R)$
- Edge: c = (u,v) in G but not in T
- There is a simple cycle containing only edge c and edges in spanning tree.
- Proof: if u is ancestor of v, result is easy. Otherwise, let l be the first node in common to paths from u to root of tree, and from v to root of tree. The paths u→v, v→l,l→u can be concatenated to form the desired cycle.



edge (I,H):
l is node G
simple cycle is (I,H,G,I)
edge (H,C):
l is node A
simple cycle is (H,C,B,A,G,H)

Property 2 of spanning trees

- Graph: G = (V,E)
- Spanning tree: T = (V,E_T,R)
- Edge: c = (u,v) in G but not in T
- There is a simple cycle Y containing only edge c and edges in spanning tree. Moreover, inserting edge c into T and deleting any edge (s→t) in Y gives another spanning tree T'.



edge (H,C):
simple cycle is (H,C,B,A,G,H)
adding (H,C) to T and deleting (A,B) gives another spanning tree

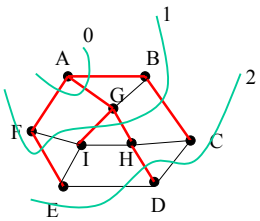
Proof of Property 2

- In T', every node is reachable from every other node.
  - Otherwise, assume node a is not reachable from node b in T'. In T, there must be a path from b to a that contains edge (s→t). In this path, replace edge (s→t) by the path in T' obtained by deleting (s→t) from cycle Y, which gives a path from b to a.

Proof of Property 2 (contd.)

- In T', there is a unique simple path between any two nodes.
  - Otherwise, suppose p1 = (x→y) and p2 = (x→y) are two distinct, edge-disjoint simple paths in T'.
  - The edge (u→v) occurs on at least one of these paths. Concatenating p1 and p2, and deleting edge (u→v) gives a path p1 in T' from u to v. This path does not contain (s→t), so it must be present in T as well. But T also contains another simple path p2 from u to v obtained by taking cycle Y and deleting (u→v) from it; this is distinct from p1 because it does contain (s→t).
  - This is a contradiction.
- Therefore, T' is a tree.

Building BFS/DFS spanning trees

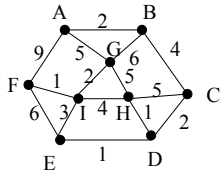


- Use sequence structure as before, but put/get edges, not nodes
  - Get edge (s,d) from structure
  - If d is not in done set,
    - add d to done set
    - (s,d) is in spanning tree
    - add out-edges (d,t) to seq structure if t is not in done set
- Example: BFS Queue
  [(dummy,A)]
  [(A,B),(A,G),(A,F)]
  [(A,G),(A,F),(B,G),(B,C)]…..

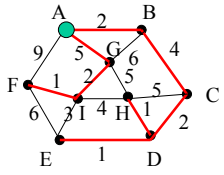
Weighted Spanning Trees

- Assume you have an undirected graph G = (V,E) with weights on each edge
- Spanning tree of graph G is tree $T = (V, E_T \subseteq E)$
  - Tree has same set of nodes
  - All tree edges are graph edges
  - Weight of spanning tree = sum of tree edge weights
- Minimal Spanning Tree (MST)
  - Any spanning tree whose weight is minimal
  - In general, a graph has several MST's
  - Applications: circuit-board routing etc.

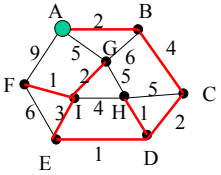
Example



Graph

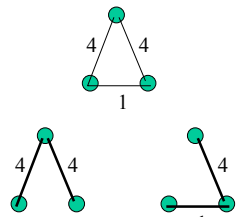


SSSP tree



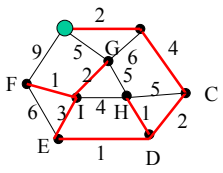
Minimal spanning tree

Caution: in general, SSSP tree is not MST



- Intuition:
  - SSSP: fixed start node
  - MST: at any point in construction, we have a bunch of nodes that we have reached, and we look at the shortest distance from any one of those nodes to a new node

Property 3 of spanning trees



Edge(G→H): 5
Cycle edges: (G→I), (I→E), (E→D),(H→D) all have weights less than (G→H)

- Graph: G = (V,E)
- Spanning tree: $T = (V, E_T, R)$
- Edge: c = (u,v) in G but not in T
- There is a simple cycle Y containing only edge c and edges in spanning tree. Moreover, weight(u→v) must be greater than or equal to weight of any edge in this cycle.
- Proof: Otherwise, modifying T by adding (u→v) and dropping heavier edge on cycle gives spanning tree of less weight.

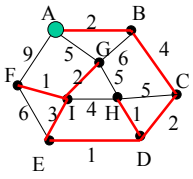
Building Minimal Spanning Trees

- Prim's algorithm: simple variation of Dijkstra's SSSP algorithm
- Change Dijkstra's algorithm so the priority of bridge (f→n) is length(f,n) rather than minDistance(f) + length(f,n)
- Algorithm produces minimal spanning tree!

Prim's MST algorithm

```
Tree MST = empty tree;
Heap h = new Heap();
//any node can be the root of the MST
h.put((dummyRoot → startNode), 0);
while (h is not empty)  {
    get minimum priority bridge (t→f);
    if (f is not lifted) {
        add (t→f) to MST;//grow MST
        make f a lifted node;
        for each edge (f→n)
            if (n is not lifted)
                h.put((f→n), length(f,n));
    }
}
```

Steps of Prim's algorithm



[((dummy→A), 0)]

[] add (dummy→A) to MST
[((A→B),2), ((A→G),5),((A→F),9)]

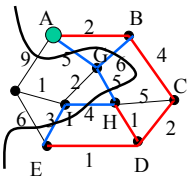
[((A→G),5),((A→F),9)] add (A→B) to MST
[((A→G),5),((A→F),9),
((B→G),6),((B→C),4)]

[((A→G),5),((A→F),9),((B→G),6)]
add (B→C) to MST
[((A→G),5),((A→F),9),((B→G),6),((C,H),5),
((C,D)2)]

……..

Property of Prim's algorithm

- At each step of the algorithm, we have a spanning tree for "lifted" nodes.
- This spanning tree grows by one new node and edge at each iteration.

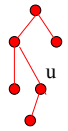


Proof of correctness

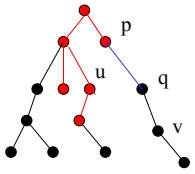
- Suppose the algorithm does not produce MST.
- Each iteration adds one new node and edge to tree.
- First iteration adds the root to tree, and at least that step is "correct".
  - "Correct" means partial spanning tree built so far can be extended to an MST.
- Suppose first k steps were correct, and then algorithm made the wrong choice.
  - Partial spanning tree P built by first k steps can be extended to an MST M
  - Step (k+1) adds edge (u→v) to P, but resulting tree cannot be extended to an MST

Proof (contd.)

- Consider simple cycle formed by adding (u→v) to M. Let p be the lowest ancestor of v in M that is also in P, and let q be p's child in M that is also an ancestor of v. So (p→q) is a bridge edge at step (k+1) as is (u→v). Since our algorithm chose (u→v) at step (k+1), weight(u→v) is less than or equal to weight(p→q).
- From Property (3), weight of (u→v) must be greater than or equal to weight(p→q).



Partial spanning tree P



Spanning tree M

Proof (contd.)

- Therefore, weight(p→q) = weight(u→v).
- This means that the tree obtained by taking M, deleting edge (p→q) and adding edge (u→v) is a minimal spanning tree as well, contradicting the assumption that there was no MST that contained the partial spanning tree obtained after step (k+1).
- Therefore, our algorithm is correct.

Complexity of algorithm

- Every edge is examines once and inserted into PQ when one of its two end points is first lifted.
- Every edge is examined again when its other end point is lifted.
- Number of insertions and deletions into PQ is |E| + 1
- Complexity = O(|E|log(|E|))

Editorial notes

- Dijkstra's algorithm and Prim's algorithm are examples of greedy algorithms:
  - making optimal choice at each step of the algorithm gives globally optimal solution
- In most problems, greedy algorithms do not yield globally optimal solutions
  - (eg) TSP
  - (eg) greedy algorithm for puzzle graph search: at each step, choose move that minimizes the number of tiles that are out of position
    - Problem: we can get stuck in "local" minima and never find the global solution