Questions/Complaints About Homework?

Here's the procedure for homework questions/complaints:

1. Read the solutions first.
2. Talk to the person who graded it (check initials)
3. If (1) and (2) don't work, talk to me.

Further comments:

- There's no statute of limitations on grade changes
  - although asking questions right away is a good strategy
- Remember that 11/12 homeworks count. Each one is roughly worth 50 points, and homework is 35% of your final grade.
  - 16 homework points = 1% on your final grade
- Remember we're grading about 100 homeworks and graders are not expected to be mind readers. It's **your** problem to write clearly.
- Don't forget to staple your homework pages together and put your name on clearly.
  - I'll deduct 2 points if that's not the case

Binary Search

Theorem: Binary search takes at most $\lfloor \log_2(n) \rfloor + 1$ loop iterations on a list of n items.

Proof: Let $P(n)$ be the statement that if $L - F = n \geq 0$, then we go through the loop at most $\lfloor \log_2(L + 1 - F) \rfloor + 1$ times.

Basis: If $L - F = 0$, then we go through the loop at most once (0 times if the $w = w_i$ is actually on the list), and $\log_2(1) + 1 = 1$.

Inductive step: Assume $P(0), \ldots, P(n)$. If $L - F = n + 1$, then either $w = w_{\lfloor (F+L)/2 \rfloor}$ (in which case we quit), or (a) $w < w_{\lfloor (F+L)/2 \rfloor}$ or (b) $w > w_{\lfloor (F+L)/2 \rfloor}$. Let L', F' be values of L and F on the next iteration.

In case (a), $L' = \lfloor (F + L)/2 \rfloor - 1$, $F' = F$, so

$$L' + 1 - F' = \lfloor (F + L)/2 \rfloor - F = \lfloor (L - F)/2 \rfloor$$

In case (b) $F' = \lfloor (F + L)/2 \rfloor + 1$, $L' = L$, so

$$L' + 1 - F' = L - \lfloor (F + L)/2 \rfloor = \lceil (L - F)/2 \rceil$$

Either way, by strong induction, it takes at most

$$1 + \lfloor \log_2(\lceil (L - F)/2 \rceil) \rfloor + 1$$

times through the loop. (One more than it takes starting at (L', F').)

A fact about the floor function:

- $1 + \lfloor x \rfloor = \lfloor 1 + x \rfloor$ for all $x \in R$

A fact about logs:

- $1 + \log_2(x/2) = 1 + \log_2(x) - \log_2(2) = \log_2(x)$

Therefore:

$$\begin{aligned} & 1 + \lfloor \log_2(\lceil (L - F)/2 \rceil) \rfloor + 1 \\ \leq\ & 1 + \lfloor \log_2((L + 1 - F)/2) \rfloor + 1 \\ =\ & \lfloor 1 + \log_2(L + 1 - F)/2) \rfloor + 1 \\ =\ & \lfloor \log_2(L + 1 - F)) \rfloor + 1 \end{aligned}$$

This is what we wanted to prove!

Bubble Sort

Suppose we wanted to sort n items. Here's one way to do it:

Input n [number of items to be sorted]
$w_1, \ldots, w_n$ [items]

Algorithm BubbleSort

for $i = 1$ to $n - 1$
  for $j = 1$ to $n - i$
    if $w_j > w_{j+1}$ **then** switch(w_j, w_{j+1}) **endif**
  endfor
endfor

Why is this right:

- Intuitively, because highest elements "bubble up" to the top

How many comparisons?

- Best case, worst case, average case all the same:
  - $(n-1) + (n-2) + \cdots + 1 = n(n-1)/2$

Proving Bubble Sort Correct

We want to show that the algorithm is correct by induction. What's the statement of the induction?

$P(k)$ is the statement that after k iterations of the outer loop, $w_{n-k+1}, \ldots, w_n$ are the k highest items, sorted in the right order.

Basis: How do we prove $P(1)$? By a nested induction!

This time, take $Q(l)$ to be the statement that after l iterations of the inner loop, w_{l+1} is higher than $\{w_1, \ldots, w_l\}$.

Basis: $Q(1)$ holds because after the first iteration of the inner loop, $w_2 > w_1$ (thanks to the switch statement).

Inductive step: After l iterations, the algorithm guarantees that $w_{l+1} > w_l$. Using the induction hypothesis, w_{l+1} is also higher than $\{w_1, \ldots, w_{l-1}\}$.

$Q(n-1)$ implies $P(1)$, so we're done with the base case of the main induction.

[**Note:** For a really careful proof, we need better notation (for value of w_l before and after the switch).]

Inductive step (for main induction): Assume $P(k)$. By the subinduction, after $n-k-1$ iterations of the inner loop, w_{n-k} is alphabetically after $\{w_1, \ldots, w_{n-(k+1)}\}$.

Combined with $P(k)$, this tells us $w_{n-k}, \ldots, w_n$ are the $k+1$ highest elements. This proves $P(k+1)$.

How to Guess What to Prove

Sometimes formulating $P(n)$ is straightforward; sometimes it's not. This is what to do:

- Compute the result in some specific cases
- Conjecture a generalization based on these cases
- Prove the correctness of your conjecture (by induction)

Example

Suppose $a_1 = 1$ and $a_n = a_{\lceil n/2 \rceil} + a_{\lfloor n/2 \rfloor}$ for $n > 1$. Find an explicit formula for a_n.

Try to see the pattern:

- $a_1 = 1$
- $a_2 = a_1 + a_1 = 1 + 1 = 2$
- $a_3 = a_2 + a_1 = 2 + 1 = 3$
- $a_4 = a_2 + a_2 = 2 + 2 = 4$

Suppose we modify the example. Now $a_1 = 3$ and $a_n = a_{\lceil n/2 \rceil} + a_{\lfloor n/2 \rfloor}$ for $n > 1$. What's the pattern?

- $a_1 = 3$
- $a_2 = a_1 + a_1 = 3 + 3 = 6$
- $a_3 = a_2 + a_1 = 6 + 3 = 9$
- $a_4 = a_2 + a_2 = 6 + 6 = 12$

$a_n = 3n$!

Theorem: If $a_1 = k$ and $a_n = a_{\lceil n/2 \rceil} + a_{\lfloor n/2 \rfloor}$ for $n > 1$, then $a_n = kn$ for $n \geq 1$.

Proof: By strong induction. Let $P(n)$ be the statement that $a_n = kn$.

Basis: $P(1)$ says that $a_1 = k$, which is true by hypothesis.

Inductive step: Assume $P(1), \ldots, P(n)$; prove $P(n+1)$.

$$\begin{aligned} a_{n+1} &= a_{\lceil (n+1)/2 \rceil} + a_{\lfloor (n+1)/2 \rfloor} \\ &= k\lceil (n+1)/2 \rceil + k\lfloor (n+1)/2 \rfloor \text{[Induction hypothesis]} \\ &= k(\lceil (n+1)/2 \rceil + \lfloor (n+1)/2 \rfloor) \\ &= k(n+1) \end{aligned}$$

We used the fact that $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$ for all n (in particular, for $n + 1$). To see this, consider two cases: n is odd and n is even.

- if n is even, $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n/2 + n/2 = n$
- if n is odd, suppose $n = 2k + 1$
    - $\lceil n/2 \rceil + \lfloor n/2 \rfloor = (k + 1) + k = 2k + 1 = n$

This proof has a (small) gap:

- We should check that $\lceil (n + 1)/2 \rceil \leq n$

In general, there is no rule for guessing the right inductive hypothesis. However, if you have a sequence of numbers

$$r_1, r_2, r_3, \ldots$$

and want to guess a general expression, here are some guidelines for trying to find the *type* of the expression (exponential, polynomial):

- Compute $\lim_{n\to\infty} r_{n+1}/r_n$
  - if it looks like $\lim_{n\to\infty} r_{n+1}/r_n = b \notin \{0,1\}$, then r_n probably has the form $Ab^n + \cdots$.
  - You can compute A by computing $\lim_{n\to\infty} r_n/b^n$
  - Try to compute the form of $\cdots$ by considering the sequence $r_n - Ab^n$; that is,

$$r_1 - Ab, r_2 - Ab^2, r_3 - Ab^3, \ldots$$

- $\lim_{n\to\infty} r_{n+1}/r_n = 1$, then r_n is most likely a polynomial.
- $\lim_{n\to\infty} r_{n+1}/r_n = 0$, then r_n may have the form $A/b^{f(n)}$, where $f(n)/n \to \infty$
  - $f(n)$ could be $n \log n$ or n^2, for example

Once you have guessed the form of r_n, prove that your guess is right by induction.

More examples

Come up with a simple formula for the sequence

$$1, 5, 13, 41, 121, 365, 1093, 3281, 9841, 29525$$

Compute limit of r_{n+1}/r_n:

$$5/1 = 5, \quad 13/5 \approx 2.6, \quad 41/13 \approx 3.2, \quad 121/41 \approx 2.95,$$
$$\ldots, 29525/9841 \approx 3.000$$

Guess: limit is 3 ($\Rightarrow r_n = A3^n + \cdot$)

Compute limit of $r_n/3^n$:

$$1/3 \approx .33, \quad 5/9 \approx .56, \quad 13/27 \approx .5, 41/81 \approx .5,$$
$$\ldots, 29525/3^{10} \approx .5000$$

Guess: limit is 1/2 ($\Rightarrow r_n = \frac{1}{2}3^n + \cdots$)+

Compute $r_n - 3^n/2$:

$$(1 - 3/2), (5 - 9/2), (13 - 27/2), (41 - 81/2), \ldots$$
$$= -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \ldots$$

Guess: general term is $3^n/2 + (-1)^n/2$

Verify (by induction ...)

One more example

Find a formula for

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)(3n+1)}$$

Some values:

- $r_1 = 1/4$
- $r_2 = 1/4 + 1/28 = 8/28 = 2/7$
- $r_3 = 1/4 + 1/28 + 1/70 = (70 + 10 + 4)/280 = 84/280 = 3/10$

Conjecture: $r_n = n/(3n+1)$. Let this be $P(n)$.

Basis: $P(1)$ says that $r_1 = 1/4$.

Inductive step:

$$\begin{aligned} r_{n+1} &= r_n + \tfrac{1}{(3n+1)(3n+4)} \\ &= \tfrac{n}{3n+1} + \tfrac{1}{(3n+1)(3n+4)} \\ &= \tfrac{n(3n+4)+1}{(3n+1)(3n+4)} \\ &= \tfrac{3n^2+4n+1}{(3n+1)(3n+4)} \\ &= \tfrac{(n+1)(3n+1)}{(3n+1)(3n+4)} \\ &= \tfrac{n+1}{3n+4} \end{aligned}$$

Faulty Inductions

Part of why I want you to write out your assumptions carefully is so that you don't get led into some standard errors.

Theorem: All women are blondes.

Proof by induction: Let $P(n)$ be the statement: For any set of n women, if at least one of them is a blonde, then all of them are.

Basis: Clearly OK.

Inductive step: Assume $P(n)$. Let's prove $P(n+1)$.

Given a set W of $n+1$ women, one of which is blonde. Let A and B be two subsets of W, each of which contains the known blonde, whose union is W.

By the induction hypothesis, each of A and B consists of all blondes. Thus, so does W. This proves $P(n) \Rightarrow P(n+1)$.

Take W to be the set of women in the world, and let $n = |W|$. Since there is clearly at least one blonde in the world, it follows that all women are blonde!

Where's the bug?

Theorem: Every integer > 1 has a unique prime factorization.

[The result is true, but the following proof is not:]

Proof: By strong induction. Let $P(n)$ be the statement that n has a unique factorization, for $n > 1$.

Basis: $P(2)$ is clearly true.

Induction step: Assume $P(2), \ldots, P(n)$. We prove $P(n+1)$. If $n+1$ is prime, we are done. If not, it factors somehow. Suppose $n+1 = rs$ $r, s > 1$. By the induction hypothesis, r has a unique factorization $\Pi_i p_i$ and s has a unique prime factorization $\Pi_j q_j$. Thus, $\Pi_i p_i \Pi_j q_j$ is a prime factorization of $n+1$, and since none of the factors of either piece can be changed, it must be unique.

What's the flaw??

Problem: Suppose $n + 1 = 36$. That is, you've proved that every number up to 36 has a unique factorization. Now you need to prove it for 36.

36 isn't prime, but $36 = 3 \times 12$. By the induction hypothesis, 12 has a unique prime factorization, say $p_1p_2p_3$. Thus, $36 = 3p_1p_2p_3$.

However, 36 is also 4×9. By the induction hypothesis, $4 = q_1q_2$ and $9 = r_1r_2$. Thus, $36 = q_1q_2r_1r_2$.

How do you know that $3p_1p_2p_3 = q_1q_2r_1r_2$.
(They do, but it doesn't follow from the induction hypothesis.)

This is a *breakdown error*. If you're trying to show something is unique, and you break it down (as we broke down $n+1$ into r and s) you have to argue that nothing changes if we break it down a different way. What if $n + 1 = tu$?

- The actual proof of this result is quite subtle

Theorem: The sum of the internal angles of a regular n-gon is $180(n-2)$ for $n \geq 3$.

Proof: By induction. Let $P(n)$ be the statement of the theorem. For $n = 3$, the result was shown in high school. Assume $P(n)$; let's prove $P(n+1)$. Given a regular $(n+1)$-gon, we can lop off one of the corners:

By induction, the sum of the internal angles of the n-gon is $180(n-2)$ degrees; the sum of the internal angles of the triangle is 180 degrees. Thus, the internal angles of the original $(n+1)$-gon is $180(n-1)$.
What's wrong??

- When you lop off a corner, you don't get a *regular* n-gon.

The fix: **Strengthen the induction hypothesis**.

- Let $P(n)$ say that the sum of the internal angles of *any* n-gon is $180(n-2)$.

Consider 0-1 sequences in which 1's may not appear consecutively, except in the rightmost two positions.

- 010110 is not allowed, but 010011 is

Prove that there are 2^n allowed sequences of length n for $n \geq 1$

Why can't this be right?

"Proof" Let $P(n)$ be the statement of the theorem.

Basis: There are 2 sequences of length 1—0 and 1—and they're both allowed.

Inductive step: Assume $P(n)$. Let's prove $P(n+1)$. Take any allowed sequence x of length n. We get a sequence of length $n+1$ by appending either a 0 or 1 at the end. In either case, it's allowed.

- If x ends with a 1, it's OK, because $x1$ is allowed to end with 2 1's.

Thus, $s_{n+1} = 2s_n = 22^n = 2^{n+1}$.

Where's the flaw?

- What if x already ends with 2 1's?

Correct expression involves separating out sequences which end in 0 and 1 (it's done in Chapter 5, but I'm not sure we'll get to it)