More Combinatorial Identities

Theorem 4: For any nonnegative integer n

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

Proof 1:

$$\begin{aligned}
& \Sigma_{k=0}^{n} k\binom{n}{k} \\
= & \Sigma_{k=1}^{n} k\frac{n!}{(n-k)!k!} \\
= & \Sigma_{k=1}^{n} \frac{n!}{(n-k)!(k-1)!} \\
= & n\,\Sigma_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \\
= & n\,\Sigma_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \\
= & n\,\Sigma_{k=0}^{n-1} \frac{(n-1)!}{(n-1-k)!k!} \\
= & n\,\Sigma_{k=0}^{n-1} C(n-1,k) \\
= & n2^{n-1}
\end{aligned}$$

Proof 2: LHS tells you all the ways of picking a subset of k elements out of n (a subcommittee) and designating one of its members as special (subcomittee chairman).

What's another way of doing this? Pick the chairman first, and then the rest of the subcommittee!

Theorem 5:

$$(n-k)\binom{n}{k} = (k+1)\binom{n}{(k+1)} = n\binom{(n-1)}{k}$$

Theorem 6:

$$C(n,k)C(n-k,j) = C(n,j)C(n-j,k)$$
$$= C(n,k+j)C(k+j,j)$$

Theorem 7: $P(n,k) = nP(n-1,k-1)$.

The Binomial Theorem

We want to compute $(x+y)^n$.
Some examples:

$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

The pattern of the coefficients is just like that in the corresponding row of Pascal's triangle!

Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Proof 1: By induction on n. $P(n)$ is the statement of the theorem.

Basis: $P(1)$ is obviously OK. (So is $P(0)$.)

Inductive step:

$$
\begin{aligned}
& (x+y)^{n+1} \\
= \ & (x+y)(x+y)^n \\
= \ & (x+y)\,\Sigma_{k=0}^{n}\binom{n}{k}x^{n-k}y^k \\
= \ & \Sigma_{k=0}^{n}\binom{n}{k}x^{n-k+1}y^k + \Sigma_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k+1} \\
= \ & \ldots \qquad\qquad \text{[Lots of missing steps]} \\
= \ & y^{n+1} + \Sigma_{k=0}^{n}(\binom{n}{k} + \binom{n}{k-1})x^{n-k+1}y^k \\
= \ & y^{n+1} + \Sigma_{k=0}^{n}(\binom{n+1}{k}x^{n+1-k}y^k \\
= \ & \Sigma_{k=0}^{n+1}\binom{n+1}{k}x^{n+1-k}y^k
\end{aligned}
$$

Proof 2: What is the coefficient of the $x^{n-k}y^k$ term in $(x+y)^n$?

Using the Binomial Theorem

Q: What is $(x+2)^4$?

A:

$$\begin{aligned} & (x+2)^4 \\ = & x^4 + C(4,1)x^3(2) + C(4,2)x^2 2^2 + C(4,3)x 2^3 + 2^4 \\ = & x^4 + 8x^3 + 24x^2 + 32x + 16 \end{aligned}$$

Q: What is $(1.02)^7$ to 4 decimal places?

A:

$$\begin{aligned} & (1+.02)^7 \\ = & 1^7 + C(7,1)1^6(.02) + C(7,2)1^5(.0004) + C(7,3)(.000008) + \cdot \\ = & 1 + .14 + .0084 + .00028 + \cdots \\ \approx & 1.14868 \\ \approx & 1.1487 \end{aligned}$$

Note that we have to go to 5 decimal places to compute the answer to 4 decimal places.

In the book they talk about the *multinomial theorem.* That's for dealing with $(x + y + z)^n$.

They also talk about the *binomial series theorem.* That's for dealing with $(x+y)^\alpha$, when α is any *real* number (like 0.3).

You're not responsible for these results.

Balls and Urns

"Balls and urns" problems are paradigmatic. Many problems can be recast as balls and urns problems, once we figure out which are the balls and which are the urns.

How many ways are there of putting b balls into u urns?

- That depends whether the balls are distinguishable and whether the urns are distinguishable

How many ways are there of putting 5 balls into 2 urns?

- If both balls and urns are distinguishable: $2^5 = 32$
  - Choose the subset of balls that goes into the first urn
  - Alternatively, for each ball, decide which urn it goes in
  - This assumes that it's OK to have 0 balls in an urn.

- If urns are distinguishable but balls aren't: 6
  - Decide how many balls go into the first urn: 0, 1, …, 5
- If balls are distinguishable but urns aren't: $2^5/2 = 16$
- If balls and urns are indistinguishable: $6/2 = 3$

What if we had 6 balls and 2 urns?

- If balls and urns are distinguishable: 2^6
- If urns are distinguishable and balls aren't: 7
- If balls are distinguishable but urns aren't:

$$2^6/2 = 2^5$$

- If balls and urns are indistinguishable: 4
  - It can't be 7/2, since that's not an integer
  - The problem is that if there are 3 balls in each urn, and you switch urns, then you get the same solution

Reducing Problems to Balls and Urns

Q1: How many different configurations are there in Towers of Hanoi with n rings?

A: The urns are the poles, the balls are the rings. Both are distinguishable.

Q2: How many solutions are there to the equation $x + y + z = 65$, if x, y, z are nonnegative integers?

A: You have 65 indistinguishable balls, and want to put them into 3 distinguishable urns (x, y, z). Each way of doing so corresponds to one solution.

Q3: How many ways can 8 electrons be assigned to 4 energy states?

A: The electrons are the balls; they're indistinguishable. The energy states are the urns; they're distinguishable.

Distinguishable Urns

How many ways can b distinguishable balls be put into u distinguishable urns?

- By the product rule, this is u^b

How many ways can b indistinguishable balls be put into u distinguishable urns?

$C(u + b - 1, b)$

Reducing Problems to Balls and Urns

Q1: How many different configurations are there in Towers of Hanoi with n rings?

A: The urns are the poles, the balls are the rings. Both are distinguishable.

- 3^n

Q2: How many solutions are there to the equation $x + y + z = 65$, if x, y, z are nonnegative integers?

A: You have 65 indistinguishable balls, and want to put them into 3 distinguishable urns (x, y, z). Each way of doing so corresponds to one solution.

- $C(67, 65) = 67 \times 33 = 2211$

Q3: How many ways can 8 electrons be assigned to 4 energy states?

A: The electrons are the balls; they're indistinguishable. The energy states are the urns; they're distinguishable.

- $C(11, 8) = (11 \times 10 \times 9)/6 = 165$

Indistinguishable Urns

How many ways can b distinguishable balls be put into u indistinguishable urns?

First view the urns as distinguishable: u^b

For every solution, look at all $u!$ permutations of the urns. That should count as one solution.

- By the Division Rule, we get: $u^b/u!$?

This can't be right! It's not an integer (e.g. $7^3/7!$).

What's wrong?

The situation is even worse when we have indistinguishable balls in indistinguishable urns. (See the book.)

Inclusion-Exclusion Rule

Remember the Sum Rule:

The Sum Rule: If there are $n(A)$ ways to do A and, distinct from them, $n(B)$ ways to do B, then the number of ways to do A *or* B is $n(A) + n(B)$.
What if the ways of doing A and B aren't distinct?

Example: If 112 students take CS280, 85 students take CS220, and 45 students take both, how many take either CS280 or CS220.

A = students taking CS280
B = students taking CS220

$$|A \cup B| = |A| + |B| - |A \cap B| = 112 + 85 - 45 = 152$$

This is best seen using a Venn diagram:

How many numbers ≤ 100 are multiples of either 2 or 5?

Let $A =$ multiples of $2 \leq 100$
Let $B =$ multiples of $5 \leq 100$

Then $A \cap B =$ multiples of $10 \leq 100$

$$|A \cup B| = |A| + |B| - |A \cap B| = 50 + 20 - 10 = 60.$$

What happens with three sets?

$$|A \cup B \cup C| =$$
$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$$
$$+|A \cap B \cap C|$$

Example: If there are 300 engineering majors, 112 take CS280, 85 takes CS 220, 95 take AEP 356, 45 take both CS280 and CS 220, 30 take both CS 280 and AEP 356, 25 take both CS 220 and AEP 356, and 5 take all 3, how many don't take any of these 3 courses?

A = students taking CS 280
B = students taking CS 220
C = students taking AEP 356

$$\begin{aligned} & |A \cup B \cup C| \\ = \; & |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| \\ & \quad + |A \cap B \cap C| \\ = \; & 112 + 85 + 95 - 45 - 30 - 25 + 5 \\ = \; & 197 \end{aligned}$$

We are interested in $\overline{A \cup B \cup C} = 300 - 197 = 103$.

The General Rule

More generally,

$$|\cup_{k=1}^n A_k| = \sum_{k=1}^{n} \sum_{\{I|I\subset\{1,\ldots,n\},\,|I|=k\}} (-1)^{k-1} |\cap_{i\in I} A_i|$$

Why is this true? Suppose $a \in \cup_{k=1}^n A_k$, and is in exactly m sets. a gets counted once on the LHS. How many times does it get counted on the RHS?

- a appears in m sets (1-way intersection)
- a appears in $C(m,2)$ 2-way intersections
- a appears in $C(m,3)$ 3-way intersections
- $\ldots$

Thus, on the RHS, a gets counted

$$\sum_{k=1}^{m} (-1)^{k-1} C(m,k) \text{ times.}$$

By the binomial theorem:

$$\begin{aligned} 0 = (-1+1)^m &= \Sigma_{k=0}^m (-1)^k 1^{m-k} C(m,k) \\ &= 1 + \Sigma_{k=1}^m (-1)^k C(m,k) \end{aligned}$$

Thus,

$$\sum_{k=1}^{m} (-1)^{k-1} C(m,k) = 1.$$

Each element in $\cup_{i=1}^k A_i$ gets counted once on both sides.

A Hard Example

Suppose $m \geq 10$. How many m-digit numbers have each of the digits 0–9 at least once? (View 00305 as a 5-digit number.)

We need a systematic way of tackling this.

Let A_j be the set of m-digit numbers that have at least one occurrence of j, for $j = 0, \ldots, 9$.

We are interested in $|A_0 \cap \ldots \cap A_9|$.

The inclusion-exclusion rule applies to unions. Can we use it?

$$\overline{A_0 \cap \ldots \cap A_9} = \overline{A_0} \cup \ldots \cup \overline{A_9}$$

$|\overline{A_i}| = 9^m$
$|\overline{A_i \cap A_j}| = 8^m$
$\ldots$

$$\begin{aligned} |\cup_{i=0}^{9} \overline{A_i}| &= 10 \times 9^m - C(10,2) \times 8^m + \cdots \\ &= \Sigma_{k=1}^{9}(-1)^{k-1}C(10,k) \times (10-k)^m \end{aligned}$$

Thus,

$$\begin{aligned} |\cap_{i=0}^{9} A_i| &= 10^m - \Sigma_{k=1}^{9}(-1)^{k-1}C(10,k) \times (10-k)^m \\ &= \Sigma_{k=0}^{9}(-1)^{k}C(10,k)(10-k)^m \end{aligned}$$

The Pigeonhole Principle

The Pigeonhole Principle: If $n+1$ pigeons are put into n holes, at least two pigeons must be in the same hole.

This seems obvious. How can it be used in combinatorial anlysis?

Q1: If you have only blue socks and brown socks in your drawer, how many do you have to pull out before you're sure to have a matching pair.

A: The socks are the pigeons and the holes are the colors. There are two holes. With three pigeons, there have to be at least two in one hole.

- What happens if we also have black socks?

Q2: Bob picks 10 numbers between 1 and 40. Alice wins if she can find two different sets of three of these numbers that have the same sum. Who wins?

A: The holes are the possible sums. The smallest sum is 6 (1+2+3), the largest is 117 (38+39+40). The pigeons are the possible ways for Alice to choose 3 numbers out of the 10 chosen by Bob.

$$\binom{10}{3} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120.$$

There's always a way for Alice to win!

Probability

Life is full of uncertainty.
Probability is the best way we currently have to quantify it.

Applications of probability arise everywhere:

- Should you guess in a multiple-choice test with five choices?
  - What if you're not penalized for guessing?
  - What if you're penalized 1/4 for every wrong answer?
  - What if you can eliminate two of the five possibilities?

- Suppose that an AIDS test guarantees 99% accuracy:
  - of every 100 people who have AIDS, the test returns positive 99 times (very few *false negative*);
  - of every 100 people who don't have AIDS, the test returns negative 99 times (very few *false positives*)

  Suppose you test positive. How likely are you to have AIDS?

  - Hint: the probability is *not* .99
- How do you compute the average-case running time of an algorithm?
- Is it worth buying a $1 lottery ticket?
  - Probability isn't enough to answer this question

(I think) everybody ought to know something about probability.